Modeling of Large Deformations of Hyperelastic Materials

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Abstract

The elastomer properties (large deformations, damping) make their use more and more common in industries such as aerospace, automotive, construction and civil engineering or even the entertainment industry... Modeling the behavior of such materials is highly nonlinear, the nonlinearities are both geometric (due to large deformations imposed) and behavioral (behavior laws used are nonlinear). This paper presents a detailed description of the numerical implementation of incompressible isotropic hyperelastic behavior. In this study, the analysis of large deformation problems of Ogden's hyperelastic is based on the finite element method (FEM) and mathematical programming. To solve the problem of balance, we suggest using the combined Newton-Raphson/Arc-length procedure. A typical example is presented to illustrate the performance of this formulation.

Keywords

Elastomers; Hyperelasticity; Large Strain; FEM; Newton-Raphson Algorithm; Arc-Length Scheme

Introduction

The theoretical studies of hyperelastic materials have mobilized many researchers for decades; see eg the work of Treloar [1], Gent [2] and Frakley [3]. However, the first important works were proposed by Mooney [4] and Rivlin [5]. Valani and al. [6] proposed to write the strain energy in a form separated following the main directions. This led to the widely used model of Ogden [7, 8]. Recently, different models have been developed for hyperelastic materials especially in biomechanics (Zulliger and al. [9]).

The objective of this paper is to present the model most widely used hyperelastic behavior and in particular the isotropic model. To do this, the different results of basic mechanics of the major changes are outlined in the first place. Are recalled and the various tensor quantities used in the modeling of continuous medium and medium formulation based on isotropic hyperelastic tensor structure. In a second step, the most widely used hyperelastic potential is exposed.

Description of Large Strains

Elastomers, with hyperelastic behavior, usually work in large strains. Note $\vec{x}$ and $\vec{X}$ positions vectors of a particle P of a deformable body in the current configuration (where the solid occupies the volume Ω) and initial (where the solid occupies the volume Ω_0). We will use the same reference $(o, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ for the initial configuration C_0 given at time t_0 and deformed configuration C_t at time t (Fig.1).

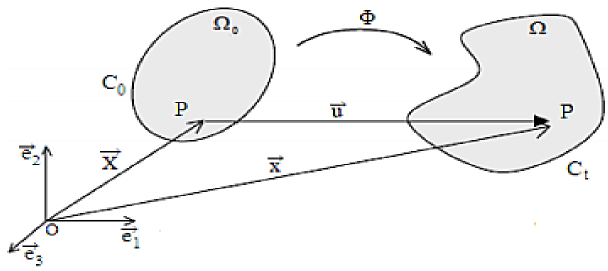

FIG. 1 INITIAL AND DEFORMED CONFIGURATIONS

For a Lagrangian description, the movement of the body can be defined relative to a reference configuration C_0, by a vector function:

$$\Phi : \begin{cases} C_0 \rightarrow C_t \\ \vec{X} \mapsto \vec{x} = \Phi(\vec{X}, t). \end{cases} \tag{1}$$

By introducing the displacement vector $\vec{u}$ and we write the equation (1) in the equivalent form:

$$\vec{x} = \vec{X} + \vec{u}(\vec{X}, t). \tag{2}$$

Order to describe the geometric transformations associated with these large strains, we introduce the deformation gradient tensor F defined by:

$$F_{ij}(\vec{X},t)=\frac{\partial \Phi_i(\vec{X},t)}{\partial X_j}=\frac{\partial x_i}{\partial X_j}=\delta_{ij}+\frac{\partial u_i}{\partial X_j}, \quad (3)$$

Where

δ_{ij} denotes the Kronecker symbol.

Or equation (3) can be written in matrix form:

$$F=I+\nabla u, \quad (4)$$

Where

I is the unit tensor,

∇ is the gradient tensor of displacements.

Due to the large displacements and large rotations, the strain tensor of Green-Lagrange ***E*** was adopted to describe the nonlinear relationship between the deformation and displacement:

$$E=\frac{1}{2}(C-I), \quad (5)$$

Where

$C=F^TF$ is the strain tensor of Cauchy-Green right.

In the case of a hyperelastic law, there is a strain energy density ***W*** which depends on one of the strain tensor and whose derivative gives the second stress tensor of Piola-Kirchhoff ***S***:

$$S_{ij}=\frac{\partial W}{\partial E_{ij}}=2\frac{\partial W}{\partial C_{ij}}. \quad (6)$$

In the particular case of isotropic hyperelasticity [10], equation (6) can be written as follows:

$$S=2\left[I_3\frac{\partial W}{\partial I_3}C^{-1}+\left(\frac{\partial W}{\partial I_1}+I_1\frac{\partial W}{\partial I_2}\right)-\frac{\partial W}{\partial I_2}C\right], \quad (7)$$

Where

I_i (i = 1, 2, 3) are the invariants of the Cauchy-Green tensor on the right, C, such as:

$$I_1=tr(C)\ ;\quad I_2=\frac{1}{2}\left[I_1^2-tr(C^2)\right]\ \text{and}\ I_3=\det(C), \quad (8)$$

Where

$tr(\bullet)$ and $\det(\bullet)$ indicate, respectively, the trace and the determinant of a tensor.

Note that ***S*** has no physical meaning but it is symmetric and purely Lagrangian, and from which one can determine the expression of the Cauchy tensor σ :

$$\sigma=\frac{1}{\det(F)}FSF^T. \quad (9)$$

For the Ogden model [7] for incompressible materials, the energy density is expressed in terms of eigenvalues λ_j (j =1, 2, 3) of the Cauchy-Green tensor right by:

$$W(\lambda_1,\lambda_2,\lambda_3)=\sum_k^m\frac{\mu_k}{\alpha_k}\left(\lambda_1^{\alpha_k}+\lambda_2^{\alpha_k}+\lambda_3^{\alpha_k}-3\right), \quad (10)$$

where the constants are the material parameters and m is the total strain energy terms.

We recall that the incompressibility of the material results in $\det(F)=1$ ou $\lambda_1\lambda_2\lambda_3=1$. Hence, the energy density can be expressed only by λ_1 and λ_2 :

$$W(\lambda_1,\lambda_2,\lambda_3)=\sum_k^m\frac{\mu_k}{\alpha_k}\left(\lambda_1^{\alpha_k}+\lambda_2^{\alpha_k}+\frac{1}{\lambda_1^{\alpha_k}\lambda_2^{\alpha_k}}-3\right). \quad (11)$$

Variational Formulation

Solving the problem balance of incompressible hyper-elastic is reduced to problem of minimizing the functional Ψ :

$$\begin{cases} Min\ \left[\Psi(u,S)=\int_\Omega\left(S.E(u)-\bar{f}.u\right)d\Omega-\int_{\Gamma_t}\bar{t}.ud\Gamma\right] \\ Subject\ to\ \begin{bmatrix} Conditions\ of\ incompressibility, \\ Boundary\ conditions\ (u=\bar{u}\ \ on\ \ \Gamma_u). \end{bmatrix} \end{cases} \quad (12)$$

Where

Ω is an isotropic structure, subject to surface forces $\bar{t}$ on the surface Γ_t,

$\bar{f}$ is the gravity forces,

$\bar{u}$ is displacements imposed on the surface Γ_u .

The nonlinear equations (12) are solved numerically with the finite element method whose the approximation of the displacement field is defined by the relation:

$$u(\mathrm{x})=N(\mathrm{x})U\ \ \text{with}\ \ \varepsilon=\nabla(N(\mathrm{x}))U, \quad (13)$$

Where

$\mathrm{x}=<\mathrm{x},\ \mathrm{y},\ \mathrm{z}>$ are the nodal coordinates,

$N(\mathrm{x})$ is the shape functions matrix,

U is the vector of unknown nodal displacements.

Therefore, the functional Ψ takes the discretized form as follows:

$$\Psi(U)=\int_{\Omega}\left(S.E(N.U)-N^{T}.\bar{f}.U\right)d\Omega-\int_{\Gamma_t}N^{T}.\bar{t}.Ud\Gamma. \qquad (14)$$

Thus, the discrete system (14) can be solved iteratively by using the combined Newton-Raphson/Arc-length procedure (see [11] for details on the implementation of this method in the context finite element).

Numerical Example

In this example, we consider a plate with a central circular hole in the stress plane and tension is sought under a constant displacement up to 100% elongation. Geometry (L × D = 25.4 × 14.24 cm²) with a unit thickness is shown in Fig. 2. For more details see reference [12]. Due to symmetry of geometry, only one quarter of the plate is modeled, as shown below.

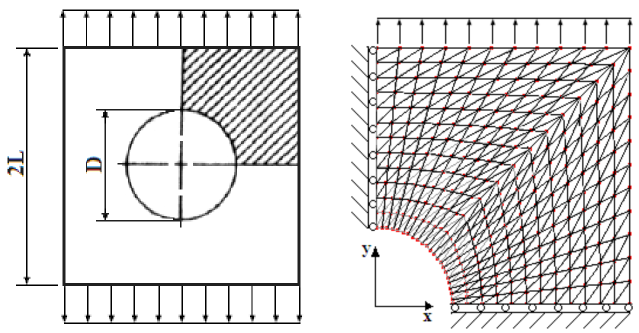

FIG. 2 GEOMETRY AND LOADING (LEFT) AND BOUNDARY CONDITIONS WITH MESHES (528 ELEMENTS T3) (RIGHT)

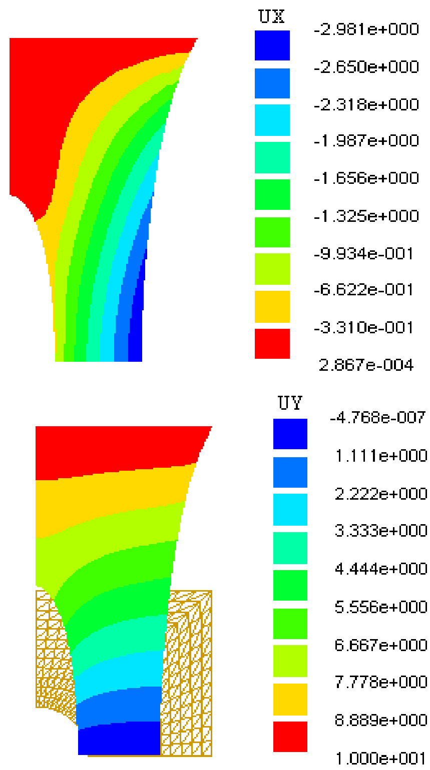

FIG. 3 DEFORMATION ALONG THE X AXIS (TOP) AND ALONG THE Y AXIS (BELOW).

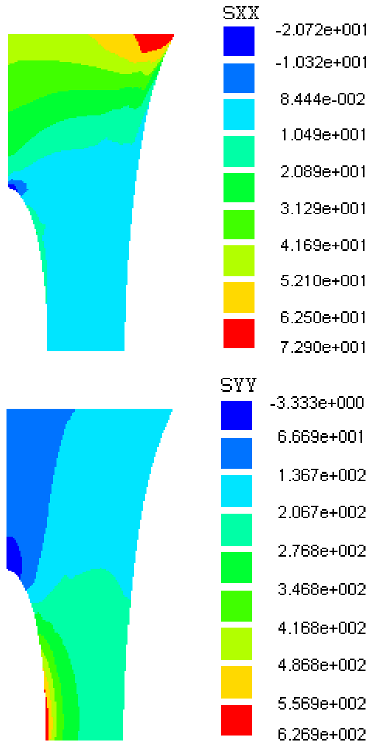

FIG. 4 ISOVALUES OF STRESS ALONG THE X AXIS (TOP) AND ALONG THE Y AXIS (BELOW).

Conclusions

A computational method by using the mathematical programming and the analysis approach hyperelastic is developed here to investigate the large deformation. A finite element model in conjunction with the combined Newton - Raphson / Arc - length procedure was adopted allowing investigating interesting applications. The obtained results illustrate the effectiveness of our approach developed.

This work can be extended in the future by taking into account other parameters as frictional contact with heat transfer.

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El Hassan Boudaia is an associate Professor at the Mechanical Engineering Department of Faculty of Science and Technology of Beni Mellal, Morocco. His Ph.D thesis in 2007 was on the subject of a meshless and finite element methods analysis for elasto-plastic contact problems with friction. His current research activities are multidisciplinary and focus mainly on design approaches and reliability analysis, especially in structural engineering. Using the elasto-plastic analysis and the meshless or the finite element method, he has published with his coworkers many works on the reliability assessment of structures, especially in the non-associated elasto-plasticity area.